Validity and truth-preservation

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Abstract

The revisionary approach to semantic paradox is commonly thought to have a somewhat uncomfortable corollary, viz. that, on pain of triviality, not all valid arguments preserve truth (Beall, 2007, 2009; Field, 2008, 2009b). We show that the standard arguments for this conclusion all break down once the *structural rule of contraction* is restricted, and we briefly rehearse some reasons for restricting such a rule.

Logical orthodoxy has it that valid arguments preserve truth (see e.g. Etchemendy, 1990; Harman, 1986, 2009):

(VTP) If an argument is valid, then, if all its premises are true, then its conclusion is also true.

Intuitive as it may seem, this claim, on natural enough interpretations of 'if' and 'true', turns out to be highly problematic. Hartry Field has argued that its most immediate justification requires all the logical and semantic resources that yield the standard semantic version of Curry's Paradox. Worse yet, both Field and Jc Beall have observed that the claim that valid arguments preserve truth almost immediately yields absurdity via Curry-like reasoning in most logics (Field, 2008; Beall, 2007, 2009). Moreover, Field has argued that by Gödel's Second Incompleteness Theorem, any semantic theory that declares all valid arguments truth-preserving must be inconsistent (Field, 2006, 2008, 2009b,a). We can't coherently require that

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valid arguments preserve truth, or so the thought goes.[1] Two main ingredients are required for this conclusion: that the conditional occurring in VTP detaches, i.e. satisfies *Modus Ponens*, and the *naïve view of truth*, viz. that (at the very least) the truth predicate must satisfy the (unrestricted) T-Scheme

$$(\textit{T-Scheme})\ Tr(\ulcorner\alpha\urcorner) \leftrightarrow \alpha,$$

where $Tr(...)$ expresses truth, and $\ulcorner\alpha\urcorner$ is a name of α.

Both assumptions lie at the heart of the leading contemporary *revisionary approaches* to semantic paradox; both *paracomplete* approaches (see e.g. Martin and Woodruff, 1975; Kripke, 1975; Brady, 2006; Field, 2003, 2007, 2008; Horsten, 2009) and *paraconsistent* ones (see e.g. Asenjo, 1966; Priest, 1979, 2006a,b; Beall, 2009). Paracomplete approaches solve paradoxes such as the Liar by assigning the Liar sentence a value in between truth and falsity, thus invalidating the Law of Excluded Middle. Paraconsistent approaches solve the Liar by taking the Liar sentence to be both true and false, avoiding absurdity by invalidating the classically and intuitionistically valid principle of *Ex Contradictione Quodlibet*. Both approaches have sought to preserve room for a detaching conditional that underwrites the T-Scheme. And when such a conditional threatens to reintroduce absurdity through Curry's Paradox, both approaches have offered a common diagnosis: they take it to show that this conditional cannot satisfy the law of contraction.

$$(\textit{Contraction})\ (\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta).$$

More generally, they require that a theory of truth be *robustly contraction free* ('rcf', for short); free, essentially, of a a conditional satisfying Contraction and other natural principles such as *Modus Ponens* (Restall, 1993).

In this paper, we assume for argument's sake the naïve view of truth, and argue that this view doesn't in fact require rejecting VTP. However, maintaining VTP requires more than revising logic so as to ensure that Contraction is no longer a theorem. Rather, it involves adopting a logic whose rules governing reasoning in the context of assumptions don't include the *structural* rule of contraction

$$(\mathsf{SContr})\ \frac{\Gamma, \alpha, \alpha \vdash \beta}{\Gamma, \alpha \vdash \beta}.$$

Once SContr is rejected, we will see, the standard objections against VTP all break down.

[1] Shapiro (2011) refers to the the claim that VTP and the naïve view of truth we introduce in the next paragraph yield triviality as the 'Field-Beall thesis'.

While few of the considerations we will raise are new, we're not aware of any detailed examination of how the challenges to VTP are affected by adopting various "substructural" logics lacking SContr.[2] In particular, it will turn out that much depends on how we conceive of what it amounts to for a conclusion to follow validly from some premises *taken jointly*.

To be sure, at least one revisionary theorist who rejects VTP holds that SContr is too basic a rule to give up (Field, 2008, pp. 10, 282-3). However, revisionary theorists have at least one powerful reason to reject such a rule. Let us assume, as is often done, that the "valid" arguments include those whose goodness depends on rules governing the truth and validity predicates (McGee, 1991; Whittle, 2004; Priest, 2006a,b; Field, 2007, 2008; Zardini, 2011). Then there exist validity-involving versions of Curry's Paradox which cannot be solved by revising the logic's *operational* rules (those governing the behavior of logical vocabulary) to ensure that the theory is robustly contraction free. This is because the only operational rules these versions of Curry's Paradox employ are a pair of rules governing a validity predicate, rules that are arguably essential to that predicate's expressing validity (Shapiro, 2011; Beall and Murzi, 2011).

The structural feature of validity encapsulated in SContr is not the only standardly accepted structural feature whose rejection would block the validity-involving versions of Curry's Paradox and allow a defense of VTP against the standard objections. An alternative "substructural" strategy, proposed by Ripley (2011), involves restricting the *transitivity* of validity as reflected in the structural rule of Cut.[3] While we will occasionally remark on this strategy, we do not have space to compare it with the strategy of giving up SContr.[4] In what follows, we will assume (as rcf theorists typically do) that validity is transitive. Likewise, we will not here be able to discuss the various ways in which one might try to make sense of and motivate the failure of SContr.[5]

The remainder of this paper is structured thus. §1 introduces the standard arguments against VTP. §2 observes that VTP follows from what we call the *naïve view of validity*, viz. that the validity predicate satisfies (generalisations of) the Rule of Necessitation and the T axiom. It then rehearses some reasons for thinking

[2] There is some relevant discussion in Shapiro (2011) and Zardini (2011).

[3] Weir (2005) also addresses semantic paradox by restricting the transitivity of validity.

[4] Both of these "substructural" approaches to semantic paradox have an advantage worth mentioning: they allow for a *unified* approach to the paradoxes of self-reference (Weir, 2005; Zardini, 2011; Ripley, 2011), as opposed to the piecemeal approach proposed by current rcf theories, where similar paradoxes, e.g. the Liar and Curry, are treated in radically different ways.

[5] For discussion of this topic, see Shapiro (2011); Zardini (2011); Beall and Murzi (2011); Mares and Paoli (2012).

that the naïve view of validity is in tension with SContr, and considers a couple of possible objections to this claim. §3 examines various possible interpretations of VTP, interpretations that become available once SContr is rejected. Specifically, it considers different ways of understanding the claim that an argument's premises are *all* true, as one finds in linear logic and what we call dual-bunching logics. It then argues that, once SContr is rejected, the standard arguments against VTP are all blocked. §4 offers some concluding remarks.

1 Three challenges to VTP

We focus on *three* challenges to VTP: that this principle's most obvious intuitive motivation rests on inconsistent premises, that VTP yields triviality via Curry-like reasoning, and that Gödel-like reasoning shows that no consistent semantic theory can endorse VTP.

1.1 The Validity Argument and Curry's Paradox

Field (2008, §2.1, §19.2) considers an argument, the Validity Argument, as he calls it, to the effect that "an inference is valid if and only if it is logically necessary that it preserves truth" (Field, 2008, p. 284). If sound, the argument for this biconditional's 'only if' direction would establish VTP. However, Field argues, it cannot be sound. Let's use $\alpha_1, ..., \alpha_n \vdash \beta$ to mean that "the argument from the premises $\alpha_1, ..., \alpha_n$ to the conclusion β is logically valid" (Field, 2008, p. 42). And let *Tr*-I and *Tr*-E, respectively, be the rules that one may infer $Tr(\ulcorner\alpha\urcorner)$ from α in any context of assumptions, and vice versa. Then Field reasons thus (we have adapted his terminology):

> *'Only if' direction*: Suppose $\alpha_1, ..., \alpha_n \vdash \beta$. Then by *Tr*-E, $Tr(\ulcorner\alpha_1\urcorner), ..., Tr(\ulcorner\alpha_n\urcorner) \vdash \beta$; and by *Tr*-I, $Tr(\ulcorner\alpha_1\urcorner), ..., Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$. By $\wedge$-E, $Tr(\ulcorner\alpha_1\urcorner) \wedge ... \wedge Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$. So by $\rightarrow$-I, $\vdash Tr(\ulcorner\alpha_1\urcorner) \wedge ... \wedge Tr(\ulcorner\alpha_n\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)$. That is, the claim that if the premises $\alpha_1, ..., \alpha_n$ are true, so is the conclusion, is valid, i.e. holds of logical necessity.
>
> *'If' direction*: Suppose $\vdash Tr(\ulcorner\alpha_1\urcorner) \wedge ... \wedge Tr(\ulcorner\alpha_n\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)$. By *Modus Ponens*, $Tr(\ulcorner\alpha_1\urcorner) \wedge ... \wedge Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$. So by $\wedge$-I, $Tr(\ulcorner\alpha_1\urcorner), ..., Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$. So by *Tr*-I, $\alpha_1, ..., \alpha_n \vdash Tr(\ulcorner\beta\urcorner)$; and by *Tr*-E, $\alpha_1, ..., \alpha_n \vdash \beta$. (Field, 2008, p. 284).[6]

[6]It may help to make Field's reasoning for the 'only if' direction explicit in natural deduction

Notice that this argument is conducted in a metalanguage containing a validity predicate (the turnstile). By contrast, no metalanguage truth predicate is explicitly used. In taking the argument to establish the thesis VTP, then, Field is assuming that the object-language sentence $Tr(\ulcorner\alpha_1\urcorner) \wedge ... \wedge Tr(\ulcorner\alpha_n\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)$ expresses the claim *that if* $\alpha_1, ..., \alpha_n$ *are all true, so is* β. We will examine this assumption closely in §3. Furthermore, he is assuming that whenever we can show that a speaker of the object-language "can validly argue to [a given claim] without using any premises," then we ourselves may assert the claim in question.

Field suggests that the Validity Argument, though it "looks thoroughly convincing at first sight" cannot be accepted, since it relies on *Tr*-I, *Tr*-E, →-I, and →-E, "which the Curry Paradox shows to be jointly inconsistent" (Field, 2008, pp. 43, 284). Let us unpack this a little. The Diagonal Lemma allows us to construct a sentence κ which, up to equivalence, intuitively says that, if it's true, then (say) you will win the lottery. Assuming that our theory of truth T is strong enough to prove the Diagonal Lemma, this means that

$$\vdash_T \kappa \leftrightarrow (Tr(\ulcorner\kappa\urcorner) \rightarrow \bot).$$

Let Π now be the following derivation of the further theorem $Tr(\ulcorner\kappa\urcorner) \rightarrow \bot$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash_T \kappa \leftrightarrow Tr(\ulcorner\kappa\urcorner) \rightarrow \bot \qquad \dfrac{Tr(\ulcorner\kappa\urcorner) \vdash_T Tr(\ulcorner\kappa\urcorner)}{Tr(\ulcorner\kappa\urcorner) \vdash_T \kappa}\,Tr\text{-E}}{Tr(\ulcorner\kappa\urcorner) \vdash_T Tr\ulcorner\kappa\urcorner \rightarrow \bot}\,{\rightarrow}\text{-E} \qquad Tr(\ulcorner\kappa\urcorner) \vdash_T Tr(\ulcorner\kappa\urcorner)}{Tr(\ulcorner\kappa\urcorner), Tr(\ulcorner\kappa\urcorner) \vdash_T \bot}\,{\rightarrow}\text{-E}}{Tr(\ulcorner\kappa\urcorner) \vdash_T \bot}\,\mathsf{SContr}}{\vdash_T Tr(\ulcorner\kappa\urcorner) \rightarrow \bot}\,{\rightarrow}\text{-I}$$

Using Π, we can then 'prove' that you will win the lottery:

$$\dfrac{\begin{matrix}\Pi \\ \vdash_T Tr(\ulcorner\kappa\urcorner) \rightarrow \bot\end{matrix} \qquad \dfrac{\dfrac{\vdash_T \kappa \leftrightarrow (Tr(\ulcorner\kappa\urcorner) \leftrightarrow \bot) \qquad \begin{matrix}\Pi \\ \vdash_T Tr(\ulcorner\kappa\urcorner) \rightarrow \bot\end{matrix}}{\vdash_T \kappa}\,{\rightarrow}\text{-E}}{\vdash_T Tr(\ulcorner\kappa\urcorner)}\,Tr\text{-I}}{\vdash_T \bot}\,{\rightarrow}\text{-E}$$

format, for the special case where we are considering an argument from the single premise α to the conclusion β. Complications raised by the multiple-premise case will be discussed in §3.

$$\dfrac{\dfrac{\dfrac{\alpha \vdash \beta \qquad \dfrac{Tr(\ulcorner\alpha\urcorner) \vdash Tr(\ulcorner\alpha\urcorner)}{Tr(\ulcorner\alpha\urcorner) \vdash \alpha}\,Tr\text{-E}}{Tr(\ulcorner\alpha\urcorner) \vdash \beta}\,\mathsf{Cut}}{Tr(\ulcorner\alpha\urcorner) \vdash Tr(\ulcorner\beta\urcorner)}\,Tr\text{-I}}{\vdash Tr(\ulcorner\alpha\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)}\,{\rightarrow}\text{-I}$$

This is the (standard) conditional-involving version of Curry's Paradox, or c-Curry, as we'll call it.[7] The derivation makes use of *Tr*-I, *Tr*-E, $\rightarrow$-I and $\rightarrow$-E, just like the Validity Argument. Hence, Field argues, one cannot accept the latter without thereby validating the former. Rcf theorists invalidate c-Curry by rejecting $\rightarrow$-I, thus resisting Π's final step (Priest, 2006b; Field, 2008; Beall, 2009; Beall and Murzi, 2011). Therefore, Field suggests, they must reject the 'only if' direction of the Validity Argument, too. (We will soon encounter an argument Field regards as valid but not truth-preserving.)

However, as Field notes, the above derivation of Curry's paradox makes use of the rule SContr. Indeed, it has long been known that Curry's paradox can be avoided by rejecting SContr (see e.g. Restall, 1994). Hence if SContr is rejected—as proposed in this context by Brady (2006), Zardini (2011), Shapiro (2011), and Beall and Murzi (2011)—the paradox no longer stands in the way of our embracing the principles used in the Validity Argument for VTP. We should note in advance that it is not clear that all types of contraction-free logics we will be considering support theories of arithmetic that prove a Diagonal Lemma. Where this is not the case, the reader should suppose that some other means of self-reference built into our semantic theory is responsible for the Curry paradoxes we will be considering. In what follows, we will ignore this complication.

Will rejecting SContr allow us to endorse the Validity Argument, then? As we will see below, matters are not this simple. Field's argument makes crucial use of rules governing the conjunction symbolized by $\wedge$. Once we no longer accept the standard structural rules, the rules for conjunction can take non-equivalent forms, and the soundness of the Validity Argument now depends on which of the available rules for $\wedge$ we accept. In §3, we will examine which of the contraction-free logics that have been proposed in response to semantic paradox underwrite the Validity Argument.[8]

[7] This terminology was introduced in Beall and Murzi (2011).

[8] Let us briefly consider how the Validity Argument fares on the alternative substructural approach that restricts transitivity. In the version of c-Curry given above, in natural deduction format, SContr is the only structural rule used. By contrast, the parallel Curry derivation in Gentzen calculus format will conclude with the following use of the structural rule of Cut

$$\frac{\vdash_T Tr(\ulcorner \kappa \urcorner) \qquad Tr(\ulcorner \kappa \urcorner) \vdash_T \bot}{\vdash_T \bot}$$

Ripley (2011) proposes a semantic theory that blocks c-Curry reasoning by invalidating Cut. His theory adds rules for *Tr* to a Gentzen calculus with entirely classical operational rules and structural rules except for Cut, which is no longer admissible in the presence of the truth rules. Weir (2005) presents a different (less classical) theory of naïve truth whose validity relation is not transitive; this shows up in his sequent-format natural deduction system as a restriction on the use of operational rules including $\rightarrow$-I and $\rightarrow$-E. We would like to make two observations about Ripley's proposal.

1.2 From VTP to absurdity via the *Modus Ponens* axiom

In addition to challenging the most obvious argument *motivating* VTP, Field offers two direct arguments according to which VTP cannot be embraced without absurdity. In the remainder of this section, then, let us examine whether we can at least *affirm* that valid arguments preserve truth. For simplicity's sake, we focus for now on arguments with only one premise. (Issues raised by multiple-premise arguments will be considered in detail in §3 below.) Moreover, we will try to affirm VTP in the object-language itself, by introducing a predicate $Val(x, y)$ which intuitively expresses that the argument from x to y is valid. VTP may now be naturally represented thus (see Beall, 2009):

$$(\mathsf{V0})\ Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner) \rightarrow (Tr(\ulcorner\alpha\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)).$$ [9]

As Field and Beall point out, V0 entails absurdity, based on principles accepted by rcf theorists (Field, 2006; Beall, 2007; Field, 2008; Beall, 2009).

Since, as we have seen, rcf theorists do not accept the rule $\rightarrow$-I, we will need two additional ingredients to obtain paradox from V0. First, the rules *Tr*-I and *Tr*-E no longer suffice; our semantic theory T needs to underwrite all instances of the T-Scheme. Second, we will use the principle that if $\vdash_T \alpha \leftrightarrow \beta$, then α and β are intersubstitutable within conditionals.[10] Given these presuppositions, V0 entails

$$(\mathsf{V1})\ Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner) \rightarrow (\alpha \rightarrow \beta).$$

Now let us assume, as rcf theorists do, that our theory T implies the validity of a single-premise version of the *Modus Ponens rule*:

$$(\mathsf{VMP})\ Val(\ulcorner(\alpha \rightarrow \beta) \wedge \alpha\urcorner, \ulcorner\beta\urcorner).$$

Hence V1 in turn entails the *Modus Ponens axiom*:

$$(\mathsf{MPA})\ (\alpha \rightarrow \beta) \wedge \alpha \rightarrow \beta.$$ [11]

On the one hand, since it retains the rule $\rightarrow$-I, it allows a defense of the Validity Argument's "only if" direction (his truth rules replace Cut in the note above), and thus of VTP. On the other hand, though Ripley's theory also endorses the *conclusion* of every instance of the Validity Argument's "if" direction, it won't allow the above intuitive *argument*, since it renders the rule $\rightarrow$-E inadmissible. See note 38 below.

[9] Strictly speaking, this should be expressed a universal generalisation on codes of sentences, but, for the sake of simplicity, we won't bother.

[10] This principle is endorsed by Field (2008, p. 253) and Beall (2009, pp. 28, 35).

[11] Following Restall (1994), this is sometimes referred to as *pseudo Modus Ponens*. See also Priest (1980), where it is described as the "counterfeit" *Modus Ponens* axiom.

However, Meyer et al. (1979) show that MPA generates Curry's Paradox. The only additional ingredient we need is the claim that it is a theorem that "conjunction is idempotent," i.e. that $\vdash \alpha \leftrightarrow \alpha \wedge \alpha$.

To see why this is so, recall that we have assumed T is strong enough to ensure $\vdash_T \kappa \leftrightarrow (Tr(\ulcorner\kappa\urcorner) \rightarrow \bot)$. Hence, given the T-Scheme and the above substitutivity principle, $\vdash_T \kappa \leftrightarrow (\kappa \rightarrow \bot)$. We can now derive absurdity starting with the relevant instance of MPA:

$$(\kappa \rightarrow \bot) \wedge \kappa \rightarrow \bot$$

Substituting κ for the equivalent $\kappa \rightarrow \bot$ gives us $\kappa \wedge \kappa \rightarrow \bot$. In view of our assumption that $\vdash_T \kappa \leftrightarrow \kappa \wedge \kappa$, another substitution of equivalents yields $\kappa \rightarrow \bot$. By substituting κ for $\kappa \rightarrow \bot$ once again, we get κ. Finally, we use $\rightarrow$-E to derive $\bot$ from $\kappa \rightarrow \bot$ together with κ.

Since VTP and VMP jointly entail the paradox-generating MPA, it would thus appear that rcf theorists cannot consistently assert that valid arguments preserve truth.[12] Field (2008, p. 377) and Beall (2009, p. 35) accept the foregoing argument, and conclude that valid arguments are not guaranteed to preserve truth: the failure of VTP is a perhaps surprising, although ultimately unavoidable, corollary of the revisionary approach to paradox, or so they argue.

1.3 From VTP to inconsistency via the Consistency Argument

A second direct argument against VTP proceeds via Gödel's Second Incompleteness Theorem, which states that no consistent theory containing a modicum of arithmetic can prove its own consistency. The argument runs thus (Field, 2006, 2008, 2009b). *If* a theory could prove that all its rules of inference preserve truth, then it could prove its own soundness, and hence its own consistency. However, we know from the Second Incompleteness Theorem that this would imply that the theory is inconsistent. Hence, on pain of the inconsistency of our arithmetic-containing semantic theory, our semantic theory can't prove that its rules of inference preserve truth after all. Yet insofar as we endorse the orthodox semantic principle VTP, we should be able to consistently add to our semantic theory an axiom stating that its rules preserve truth (see Field, 2009a, p. 351n10).

To pose this challenge to VTP, Field considers what he calls the Consistency Argument (Field, 2006). This is an argument which, one might think, one should be able to run *within* any theory T containing a 'universal' truth predicate satisfying the unrestricted T-Scheme. The argument is in two steps (Field, 2008, p. 286):

[12] See Beall (2007), Beall (2009, pp. 34-41), Shapiro (2011, p. 341) and Beall and Murzi (2011).

(i) one inductively proves, within T, that T is sound, and

(ii) one proceeds to argue, again within T, that T must therefore be consistent.

Though intuitively sound, the Consistency Argument must fail if T is to be consistent. Field's claim is that its failure must be blamed on a failure of VTP. Let us unpack his reasoning. He observes that (ii) cannot be problematic for those *paracomplete* theorists, including himself, who hold that "inconsistencies imply everything". As he points out, the target theories "certainly imply $\neg Tr(\ulcorner 0 = 1 \urcorner)$, so the soundness of T would imply that '0 = 1' isn't a theorem of T; and this implies that T is consistent." Moreover, (ii) will also be unproblematic for any *paraconsistent* theorist who holds that an adequate semantic theory must imply the universal generalization over instances of the schema $\neg Tr(\ulcorner \alpha \wedge \neg\alpha \urcorner)$. In this case as well, if T could prove its soundness, it could thereby prove its consistency.[13]

Field therefore concludes that the problem with the Consistency Argument lies with (i). The subargument alluded to in (i) can be divided into three steps (Field, 2008, p. 287):

(1) Each axiom of T is true;

(2) Each rule of inference of T preserves truth, in the sense of VTP: whenever its premises are true, so is the conclusion;

(3) Hence all theorems of T are true, i.e. T is sound.

Field argues persuasively that good enough choices of T will be able to establish (1), and will be able to derive (3) from (1) and (2). Hence, he concludes, "[t]he only place that the argument can conceivably go wrong is ... in (2)." This conclusion is endorsed by Beall (2009, pp. 115-6).

In sum, VTP must fail for at least two reasons, or so contemporary revisionary wisdom goes. As Beall writes: "such a claim ... needs to be rejected, and I reject it" (Beall, 2009, p. 35).

2 Naïve validity and Validity Curry

What role, then, if any, is left for the notion of validity, if valid arguments are not guaranteed to preserve truth? Field (2008, 2009b, 2010) suggests that validity normatively constrains belief: very roughly, one shouldn't fully believe the premises of a valid argument without fully believing its conclusion. We take no position here

[13]See Field (2006, pp. 593-5).

on whether the role of the notion of validity can be explained without recourse to truth-preservation.[14] Instead, we'll suggest in the remainder of this paper that revisionary theorists *need not* and *should not* reject VTP.

2.1 Naïve validity

Still restricting our attention to single-premise arguments, consider the following two principles for the use of the validity predicate: that, if one can derive ψ from ϕ, one can derive on no assumptions that the argument from ϕ to ψ is valid, and that, from ϕ and the claim that the argument from ϕ to ψ is valid, one can infer ψ.[15]

Both rules are highly intuitive. If $Val(x, y)$ expresses *validity*, it seems natural to assume that an adequate semantic theory T must include the following introduction rule for $Val(x, y)$, which, by analogy with $\rightarrow$-I or Conditional Proof, we'll call *Validity Proof*:

$$\text{(VP)}\ \frac{\alpha \vdash_T \beta}{\vdash_T Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner)}\ .$$

If T's rules are valid, and we can derive β from α in T, then T must be able to assert the sentence $Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner)$, expressing that the argument from α to β is valid. But it also seems natural to assume that T contains an elimination rule for $Val(x, y)$, which we'll call *Validity Detachment*:

$$\text{(VD)}\ \frac{\Gamma \vdash_T Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner) \qquad \Delta \vdash_T \alpha}{\Gamma, \Delta \vdash_T \beta}\ .$$

If, from a given context of assumptions, we can derive in T the sentence α and from another context we can derive that the argument from α to β is valid, then it must be possible (from the assumptions taken together) to derive β.[16]

[14] For the record, we think that even if VTP holds, an explanation of the role of the notion of validity will have to involve normative considerations such as those Field advances.

[15] To the best of our knowledge, these rules are first discussed in Priest (2010). For further discussion, see Beall and Murzi (2011) and Murzi (2011). Shapiro (2011) proposes introducing a validity predicate governed by the equivalences $Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner) \dashv\vdash_T \alpha \Rightarrow \beta$, where $\Rightarrow$ is an *entailment connective* whose introduction and elimination rules in turn render VP and VD derivable. Such a connective is common in the tradition of relevant and paraconsistent logic: see e.g. Anderson and Belnap (1975, p. 7) and Priest and Routley (1982).

[16] We have written the rule VP without side assumptions. That is because the acceptability of a version including side assumptions

$$\text{(VP}^*\text{)}\ \frac{\Gamma, \alpha \vdash_T \beta}{\Gamma \vdash_T Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner)}$$

depends on the properties of the structural comma. For example, if the comma obeys *weakening* and we get $\beta, \alpha \vdash_T \beta$, then VP* allows us to derive $\beta \vdash_T Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner)$. But where β is contingent, it shouldn't follow from β that it is entailed by any sentence. A similar problem arises if the

The rules VP and VD can also be viewed as *generalizations* of natural rules for a predicate that expresses logical truth: namely, analogues of the rule of Necessitation and of a rule corresponding to the T axiom. To see this, it is sufficient to instantiate VP and VD using a constant T expressing logical truth. Instantiating VP yields a notational variant of Necessitation, rewritten using our two place predicate $Val(x, y)$ in place of a necessity operator:

$$(\mathsf{NEC}^*)\ \frac{\mathsf{T} \vdash_T \beta}{\vdash_T Val(\ulcorner \mathsf{T} \urcorner, \ulcorner \beta \urcorner)}.$$

Likewise, instantiating VD thus

$$\frac{\Gamma \vdash_T Val(\ulcorner \mathsf{T} \urcorner, \ulcorner \beta \urcorner) \qquad \mathsf{T} \vdash_T \mathsf{T}}{\Gamma, \mathsf{T} \vdash_T \beta}$$

yields a notational variant of a rule corresponding to the T axiom for a necessity operator:

$$(\mathsf{T}^*)\ Val(\ulcorner \mathsf{T} \urcorner, \ulcorner \beta \urcorner), \mathsf{T} \vdash_T \beta.$$

The intuitiveness of our rules VP and VD is thus underscored by the close connection they underwrite between the behavior of a predicate expressing logical truth and the behavior of an operator expressing logical necessity.

We will therefore call the view that 'valid' satisfies VP and VD the *naïve view of validity* (Murzi, 2011). One first point that deserves emphasis is that, on the naïve truth of truth we've assumed at the beginning of this paper, such a view entails V0, our object-language statement of VTP for single-premise arguments. This can be shown using what is essentially a version of Field's Validity Argument, except that the validity of the argument from α to β is now expressed using an object-language predicate rather than using a turnstile in the metalanguage:[17]

$$\dfrac{\dfrac{\dfrac{\dfrac{Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner) \vdash_T Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner) \qquad \dfrac{Tr(\ulcorner\alpha\urcorner) \vdash_T Tr(\ulcorner\alpha\urcorner)}{Tr(\ulcorner\alpha\urcorner) \vdash_T \alpha}\,Tr\text{-E}}{Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner), Tr(\ulcorner\alpha\urcorner) \vdash_T \beta}\,\mathsf{VD}}{Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner), Tr(\ulcorner\alpha\urcorner) \vdash_T Tr(\ulcorner\beta\urcorner)}\,Tr\text{-I}}{Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner) \vdash_T Tr(\ulcorner\alpha\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)}\,\rightarrow\text{-I}}{\vdash_T Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner) \rightarrow (Tr(\ulcorner\alpha\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner))}\,\rightarrow\text{-I}$$

comma obeys *exchange*. From VD and Cut we get $Val(\ulcorner\alpha\urcorner,\ulcorner\alpha\urcorner), \alpha \vdash_T \alpha$, whence exchange yields $\alpha, Val(\ulcorner\alpha\urcorner,\ulcorner\alpha\urcorner) \vdash_T \alpha$ and VP* allows us to derive $\alpha \vdash_T Val(\ulcorner Val(\ulcorner\alpha\urcorner,\ulcorner\alpha\urcorner)\urcorner,\ulcorner\alpha\urcorner)$. But if α is contingent, it shouldn't follow from α that it is entailed by a logical truth. Zardini (2012), whose comma obeys both weakening and exchange, avoids these problems by restricting the side assumptions in VP$*$ to logical compounds of *validity claims*. See also Priest and Routley (1982).

[17] Ripley (2011) offers a similar defense of VTP, using VP and the sequent $\alpha, Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner) \vdash_T \beta$. Shapiro (2011) explains that on the version of the naïve view presented there (see note 15 above), $Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner)$ implies $Tr(\ulcorner\alpha\urcorner) \Rightarrow Tr(\ulcorner\beta\urcorner)$.

A second point to notice is that, natural though they may seem, VP and VD lead us into trouble—which should of course be expected, since NEC* and T* are nothing but the key ingredients of the Myhill-Kaplan-Montague Paradox, or Paradox of the Knower (Myhill, 1960; Kaplan and Montague, 1960; Murzi, 2011).[18]

2.2 Validity Curry

The Diagonal Lemma allows us to construct a sentence π, which intuitively says of itself, up to equivalence, that it validly entails that you will win the lottery:

$$\vdash_T \pi \leftrightarrow Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner).$$

We may then 'prove' π as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi \vdash_T \pi \qquad \vdash_T \pi \leftrightarrow Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)}{\pi \vdash_T Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)}\,\rightarrow\text{-E} \qquad \pi \vdash_T \pi}{\pi, \pi \vdash_T \bot}}{\pi \vdash_T \bot}\,\textsf{SContr}}{\vdash_T Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)}\,\textsf{VP} \qquad \vdash_T \pi \leftrightarrow Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)}{\vdash_T \pi}$$

One application of VD finally allows us to conclude $\vdash_T \bot$ from this and $\vdash_T Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)$. Our revisionary theory of truth and validity, T, proves on no assumptions that you will win the lottery.[19] Call this the Validity Curry, or v-Curry, for short, to contrast it with the standard conditional-involving version of Curry's Paradox, or c-Curry.[20] Rcf theorists invalidate c-Curry by rejecting $\rightarrow$-I (Priest, 2006b; Field, 2008; Beall, 2009; Beall and Murzi, 2011). Unlike c-Curry, however, the v-Curry Paradox makes no use of $\rightarrow$-I, and hence it cannot be invalidated by rejecting such a rule. On the other hand, the above derivation of v-Curry presupposes SContr (Beall and Murzi, 2011). Hence if VP and VD hold, there is only one

[18] Shapiro (2011) identifies two challenges to the naïve view: a "direct argument" that it leads straight to paradox, and an "indirect argument" that it entails a version of the paradox-producing VTP.

[19] To the best of our knowledge, the first known occurrence of the Validity Curry is in the 16th-century author Jean de Celaya. See Read (2001, fn. 11-12) and references therein. Albert of Saxony discusses a contrapositive version of the paradox in his *Insolubles* (Read, 2010, p. 211). A more recent version can be found in Priest and Routley (1982), and surfaces again in Whittle (2004, fn. 3), Clark (2007, pp. 234-5) and Shapiro (2011, fn. 29). For a first comprehensive discussion of the Validity Curry, see Beall and Murzi (2011). For a defence of the claim that Validity Curry is a genuine semantic paradox, see §2.3 below and Murzi (2011).

[20] This terminology was first introduced in Beall and Murzi (2011). Ultimately, however, the distinction in terms of predicate versus connective may not be the essential one. Whittle (2004) and Shapiro (2011) discuss a version of Curry's Paradox, involving a "consequence connective" or "entailment connective," which poses much the same challenge to rcf theorists as does v-Curry.

revisionary way out of the v-Curry Paradox, viz. rejecting SContr, thus adopting a *substructural* logic—a logic where some of the standardly accepted structural rules fail (Shapiro, 2011; Beall and Murzi, 2011; Murzi, 2011; Zardini, 2011).[21]

Before examining in §3 how rejecting SContr affects VTP and the Validity Argument, we'd first like to offer a partial defence of our claim that v-Curry Paradox is a reason for revisionary logician to adopt a substructural logic. To this end, we'll consider in the next section two natural responses to the claim that the Validity Curry is a genuine semantic paradox, and offer replies on the substructural logician's behalf.

2.3 A genuine semantic paradox

If the v-Curry Paradox isn't a genuine semantic paradox, one of VP and VD must not unrestrictedly hold. As it turns out, there are *prima facie* compelling reasons for restricting both.[22] One argument against VP runs thus. One simply notices that the subproof in the v-Curry derivation relies on a substitution instance of the *logically invalid* biconditional proved by the Diagonal Lemma, viz. $\pi \leftrightarrow Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)$, and hence isn't logically valid, contrary to what an application of VP at the end of the subproof would require. Furthermore, in both versions of the v-Curry Paradox, VD gets used in the subderivation, and, it might be objected, surely such a rule isn't logical. More precisely, Roy Cook (2012) has argued that the T-Scheme isn't logically valid, if by logical validity one means truth under all uniform interpretations of the non-logical vocabulary. Cook's reasoning would apply equally to the status of VP and VD.[23]

These objections have an important virtue: they help us understand what the v-Curry Paradox really is a paradox of. More precisely, they show that the v-Curry Paradox is not paradox of *purely logical*, or *interpretational*, in John Etchemendy's term, validity (Etchemendy, 1990).[24] Indeed, a recent result by Jeff Ketland shows

[21]For an early anticipation of the argument from naïve validity to the rejection of SContr (in the form of multiple discharge of assumptions), see Priest and Routley (1982). Priest and Routley, whose entailment connective obeys analogues of VP and VD, discuss several resulting paradoxes which they blame on the "suppression of innocent premises." By contrast, Ripley (2011) blocks v-Curry at the final step using VD, which is inadmissible in his nontransitive theory for the same reason that $\rightarrow$-E is inadmissible. See note 38 below.

[22]Thanks to Roy Cook and Jeff Ketland for raising these potential concerns.

[23]Field (2008, §20.4) himself advances versions of this line of argument, while discussing what is in effect a validity-involving version of the Knower Paradox resting on NEC* and T*. See especially Field (2008, p. 304 and p. 306). On the question whether his conception of the extension of the validity predicate consistently allows him to do so, see note 25 below.

[24]Here we take the logical vocabulary to be the standard vocabulary of some first-order, perhaps non-classical, logic.

that purely logical validity *cannot* be paradoxical. Ketland (2012) proves that Peano Arithmetic (PA) can be conservatively extended by means of a predicate expressing logical validity, governed by intuitive principles that are themselves derivable in PA. Thus, purely logical validity is a consistent notion if PA is consistent, which should be enough to warrant belief that purely logical validity simply *is* consistent.

However, it seems to us that there are broader notions of validity than purely logical validity. Thus, neither of the above objections applies to versions of the v-Curry Paradox in which 'valid' expresses *representational* validity, whereby (roughly) validity is equated with preservation of truth in all possible circumstances (Read, 1988; Etchemendy, 1990; McGee, 1991). But VP, VD and the arithmetic required to prove the Diagonal Lemma are, at least intuitively, valid in this sense. Nor does the objection apply to conceptions of validity which take 'valid' to express the consequence relation of one's semantic theory, provided that the naïve validity rules and enough arithmetic are part of that relation.[25] Insofar as VD is valid in one of these broader senses, and insofar as the VP and VD govern the use a predicate expressing validity in that sense, there is at least one—important—reading of 'valid' on which the use of VP in the v-Curry derivation is sound. The v-Curry Paradox is a paradox of *validity*, not *purely logical* validity.[26]

To be sure, one might instead either reject VP on different grounds, or perhaps reject VD. One natural enough argument against the latter rule runs thus. Suppose validity is recursive. Then, one might argue, T^*, and hence VD, must fail. For, if validity is recursively enumerable, an argument is valid if and only if its conclusion can be derived from its premises in some recursively axiomatisable theory T. That is, the validity predicate $Val(x, y)$ is just a notational variant of $Prov_T(x, y)$, where this expresses that there is a T-derivation of y from x. Yet, the argument continues, we know from Löb's Theorem that, if T contains enough arithmetic (if it proves the so-called derivability conditions), T cannot contain, on pain of triviality, all instances of the provability-in-T analogue of T^*, $Prov_T(\ulcorner \top \urcorner, \ulcorner \alpha \urcorner) \rightarrow \alpha$. Hence, one might conclude, T may not contain all instances of T^* either, and hence of VD, *a*

[25] In recent unpublished work, Cook in fact shows how this response can be strengthened: it is possible to formulate a modified Validity Curry paradox in such a way that the arithmetic necessary to prove the Diagonal Lemma need not be included in the scope of the validity relation.

[26] It might be objected that validity simply *is* purely logical validity, and that the uses of 'valid' we are introducing are unnatural ones. However, while this may be a legitimate *reaction* to the v-Curry Paradox, it is worth pointing out that several semantic theorists, including rcf theorists such as Field and Priest, resort to non purely logical notions of validity. For instance, Field (2007, 2008) extensionally identifies validity with, essentially, preservation of truth in all ZFC models of a certain kind, thus taking validity to (wildly) exceed purely logical validity. It seems to us that this use of 'valid' is in tension with the purely logical sense Field (2008) appeals to at p. 304 and especially p. 306. Likewise, McGee (1991, p. 43-9) takes logical necessity to extend to arithmetic and truth-theoretic principles.

fortiori.

We find this conclusion problematic—it seems to us that rejecting VD, or VP, for that matter, isn't really a comfortable option for proponents of the naïve view of truth. In a nutshell, together with the naïve view of truth, the naïve view of validity is but an instance of the general thought underpinning the revisionary approach to paradox—what we may call the *naïve semantic properties*. This is the view that one cannot revise naïve semantic principles without thereby also revising naïve semantic properties, and that, on pain of triviality, semantic properties should be held fixed, and *logic* must change. Arguably, the naïve view of semantic properties has it that validity *is* factive, and that we, and hence our semantic theory, must be able to say so, on pain of not being able to consistently assert what we know to be true. If T does indeed meet the conditions for Löb's Theorem, we would like to suggest, then the correct reaction to the objection is instead to concede that $Val(x, y)$ can't be replaced with $Prov_T(x, y)$, and hence that that naïve validity is not recursively enumerable.[27]

It might be objected that we could revise, or refine, our naïve conception of validity, which is after all naïve (McGee, 1991, p. 45). But, then, a parallel argument would show that, when faced with the Liar Paradox, the c-Curry Paradox, and other paradoxes of truth, we should similarly revise our conception of *truth*, which is precisely what proponents of the naïve view of semantic properties take to be the *wrong* response to semantic paradox. For the time being, we'll assume that the Validity Curry is a genuine semantic paradox, and that giving up SContr, as suggested in Shapiro (2011) and Zardini (2011), is a legitimate revisionary response to it, and to semantic paradoxes more generally. We shall now argue that, on this admittedly controversial assumption, of which we've only offered a partial defence, all three arguments against VTP break down.

3 Validity and truth-preservation

All three challenges to VTP turn out to rest crucially on how our object-language expresses validity and truth-preservation for arguments with multiple premises. First, recall that Field seeks to undermine VTP by arguing that we can't, on pain

[27]We don't have space to expand on this point here. Priest (2006b, §3.2) argues at length that the "naïve notion of proof" is recursive, whence naïve provability, a species of naïve validity, is recursively enumerable. Here we simply notice that his arguments are consistent with the view that naïve *validity* isn't. Finally, we'd like to point out that some SContr-free semantic theories extending contraction-free arithmetics may not be strong enough to satisfy Löb's Theorem's applicability conditions, in which case the objection from Löb's Theorem we are considering would not apply in the first place.

of paradox, accept the argument which naturally motivates it. But, as we have pointed out, this argument (the Validity Argument) presupposes that the truth-preservingness of an inference from $\alpha_1, ..., \alpha_n$ to β can be expressed using the object-language sentence $Tr(\ulcorner\alpha_1\urcorner) \wedge ... \wedge Tr(\ulcorner\alpha_n\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)$. Second, the argument from VTP to PMP and absurdity used the simplifying assumption that the validity of the two-premise *Modus Ponens* rule can be expressed using a single-premise validity predicate as $Val(\ulcorner\alpha \wedge (\alpha \rightarrow \beta)\urcorner, \ulcorner\beta\urcorner)$. Finally, spelling out the Consistency Argument requires expressing in the object-language the claim that each of our semantic theory T's rules of inference preserves truth, where these will include multi-premise rules such as *Modus Ponens*.

3.1 Premise-aggregating connectives

We will therefore assume that truth-preservation and validity for arguments with a finite number of premises can be expressed using some "premise-aggregating connective" $\odot$, in the following ways:[28]

(a) The claim that the argument from premises $\alpha_1, ..., \alpha_n$, taken together, to conclusion β preserves truth can be expressed in the object-language as $Tr(\ulcorner\alpha_1\urcorner) \odot ... \odot Tr(\ulcorner\alpha_n\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)$.

(b) The claim that the argument from premises $\alpha_1, ..., \alpha_n$, taken together, to conclusion β is valid can be expressed using the object-language's binary validity predicate as $Val(\ulcorner\alpha_1 \odot ... \odot \alpha_n\urcorner, \ulcorner\beta\urcorner)$.

Our question now becomes this. Is there an understanding of the logical behavior of $\odot$ on which (a) and (b) are true, but each of our three challenges to VTP is blocked?

Before examining the three challenges in turn, we now consider the chief options for the rules governing $\odot$ in the context of a substructural natural deduction system. For the time being, we will work within a structural framework in which the "taking together" of assumptions—what we have been indicating using commas to the left of the turnstile—can be represented by means of "multisets." These are structures that behave like sets except for the fact that they keep track of the number of occurrences of each member (Meyer and McRobbie, 1982a,b). Using multisets rather than (e.g.) sequences renders redundant Gentzen's structural rule of *exchange*:

[28]For arguments with an infinite number of premises, we will need universal quantification to express truth-preservation. None of the objection against VTP we will consider, however, depend on consideration of infinite-premise arguments.

$$(\text{SExch})\ \frac{\Gamma, \alpha, \beta \vdash \gamma}{\Gamma, \beta, \alpha \vdash \gamma}.$$

By contrast, SContr is not redundant, and neither is the structural rule of *weakening*:

$$(\text{SWeak})\ \frac{\Gamma, \alpha \vdash \gamma}{\Gamma, \beta, \alpha \vdash \gamma}.$$

Indeed, once one or more of these structural rules is rejected, one can formulate operational rules for two different connectives, rules that become equivalent only in the presence of both SContr and SWeak. These are the rules that govern, respectively, the "multiplicative" and "additive" conjunctions of linear logic, a multiset-based logic in which both SWeak and SContr are rejected (Girard, 1987).[29]

$$(\otimes\text{-I})\ \frac{\Gamma \vdash \alpha \qquad \Delta \vdash \beta}{\Gamma, \Delta \vdash \alpha \otimes \beta} \qquad (\otimes\text{-E})\ \frac{\Gamma, \alpha, \beta \vdash \gamma \qquad \Delta \vdash \alpha \otimes \beta}{\Gamma, \Delta \vdash \gamma}$$

$$(\&\text{-I})\ \frac{\Gamma \vdash \alpha \qquad \Gamma \vdash \beta}{\Gamma \vdash \alpha \,\&\, \beta} \qquad (\&\text{-E1})\ \frac{\Gamma \vdash \alpha \,\&\, \beta}{\Gamma \vdash \alpha} \qquad (\&\text{-E2})\ \frac{\Gamma \vdash \alpha \,\&\, \beta}{\Gamma \vdash \beta}$$

Since it will prove important later, we note that the structural comma appears in the rules for $\otimes$, whereas it does not appear in the rules for &. In the terminology of Belnap (1982, 1993), the additive rules are "structure-free" while the multiplicative rules are "structure-dependent". Finally, in this structural setting, our assumption of the transitivity of validity can be codified in terms of the following version of the cut rule:

$$(\text{Cut})\ \frac{\Gamma \vdash \alpha \qquad \Delta, \alpha \vdash \beta}{\Delta, \Gamma \vdash \beta}$$

3.2 The Validity Argument

The first point we would like to make is that, in the absence of SContr, the 'only if' direction of the Validity Argument (the direction that would establish VTP) fails when the premise-aggregating connective $\odot$ is construed as the additive & in a multiset-based logic.

To see why, consider again the reasoning Field challenges, rewritten with premise-aggregation expressed using &:

> Suppose $\alpha_1, ..., \alpha_n \vdash \beta$. Then by *Tr*-E, $Tr(\ulcorner \alpha_1 \urcorner), ..., Tr(\ulcorner \alpha_n \urcorner) \vdash \beta$; and by *Tr*-I, $Tr(\ulcorner \alpha_1 \urcorner), ..., Tr(\ulcorner \alpha_n \urcorner) \vdash Tr(\ulcorner \beta \urcorner)$. By & -E, $Tr(\ulcorner \alpha_1 \urcorner)\ \&\ ...\ \&\ Tr(\ulcorner \alpha_n \urcorner) \vdash Tr(\ulcorner \beta \urcorner)$. So by $\rightarrow$-I, $\vdash Tr(\ulcorner \alpha_1 \urcorner)\ \&\ ...\ \&\ Tr(\ulcorner \alpha_n \urcorner) \rightarrow Tr(\ulcorner \beta \urcorner)$.

[29] While linear logic is standardly presented in Gentzen calculus format, the above natural deduction rules appear in Troelstra (1992, p. 57) and O'Hearn and Pym (1999).

In deriving $Tr(\ulcorner\alpha_1\urcorner)$ & ... & $Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$ from $Tr(\alpha_1), ..., Tr(\alpha_n) \vdash Tr(\beta)$, this reasoning requires $n-1$ uses of the inference pattern

$$(\&\text{-L})\ \frac{\Gamma, \alpha_1, \alpha_2 \vdash \beta}{\Gamma, \alpha_1 \,\&\, \alpha_2 \vdash \beta}\,.$$

In the formulation we have adapted from Field, that inference is justified by appeal to &-E. Indeed, in the presence of SContr, either of our twin elimination rules rules &-E1 and &-E2 yields &-L. Here is a derivation using &-E2, SContr, Cut, and the reflexivity of validity:

$$\dfrac{\dfrac{\dfrac{\Gamma, \alpha_1, \alpha_2 \vdash \beta \qquad \dfrac{\alpha_1 \,\&\, \alpha_2 \vdash \alpha_1 \,\&\, \alpha_2}{\alpha_1 \,\&\, \alpha_2 \vdash \alpha_2}\,\&\text{-E2}}{\Gamma, \alpha_1, \alpha_1 \,\&\, \alpha_2 \vdash \beta}\,\mathsf{Cut} \qquad \alpha_1 \,\&\, \alpha_2 \vdash \alpha_1}{\Gamma, \alpha_1 \,\&\, \alpha_2, \alpha_1 \,\&\, \alpha_2 \vdash \beta}\,\mathsf{Cut}}{\Gamma, \alpha_1 \,\&\, \alpha_2 \vdash \beta}\,\mathsf{SContr}$$

In a logic without SContr, on the other hand, &-L fails. Accordingly, if we defend the Validity Argument against Field's objection from c-Curry by rejecting SContr, the argument's "only if" direction will still fail to be sound as long as the premise-aggregating connective $\odot$ is construed as &. Moreover, matters are no different if we accept SWeak, thus replacing linear logic with what is known as an "affine" logic.[30]

Hence, insofar as one wishes to preserve the Validity Argument while rejecting SCont (and thus avoiding c-Curry and v-Curry), one ought not interpret the premise-aggregating $\odot$ as the additive conjunction & of a multiset-based logic. On the other hand, both directions of the Validity Argument go through, even in the absence of SContr, provided that $\odot$ is construed as the multiplicative $\otimes$ of such a logic. For given $\alpha_1 \otimes \alpha_2 \vdash \alpha_1 \otimes \alpha_2$, the rule $\otimes$-E immediately yields

$$(\otimes\text{-L})\ \frac{\Gamma, \alpha_1, \alpha_2 \vdash \beta}{\Gamma, \alpha_1 \otimes \alpha_2 \vdash \beta}\,.$$[31]

Then $n-1$ uses of this inference let us establish that $\alpha_1, ..., \alpha_n \vdash_T \beta$ only if $\vdash_T Tr(\ulcorner\alpha_1\urcorner) \otimes ... \otimes Tr(\ulcorner\alpha_n\urcorner) \rightarrow \beta$. Indeed, with $\otimes$ as premise-aggregating connective, Elia Zardini (2011) has recently proved a generalization of the "only if" conclusion of Field's Validity Argument, using essentially the reasoning sketched by Field

[30] In that case, however, the "if" direction of the Validity Argument *will* go through for & as premise-aggregating connective. Deriving $Tr(\ulcorner\alpha_1\urcorner), ..., Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$ from $Tr(\ulcorner\alpha_1\urcorner)$ & ... & $Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$ requires the inverse of &-L, which obtains in the presence of SWeak.

[31] In single-conclusion Gentzen calculus formulations (which suffice for our purposes, as our derivations all involve the language's negation-free fragment), the connective $\otimes$ is governed by the twin rules $\otimes$-I and $\otimes$-L.

above.[32] And the "if" direction is no harder to establish. Summarizing, we can say that Field's objection to the "only if" direction of the Validity Argument fails when semantic theory is based on an underlying logic that lacks SContr, as long as this logic is multiset-based and we state the argument's conclusion using multiplicative conjunction.

Furthermore, a multiset-based logic is not the only way to implement the strategy of vindicating the Validity Argument, while avoiding c-Curry and v-Curry, by rejecting a structural contraction rule. A second way is to use one of the substructural logics in which premises can be aggregated (or "taken together') in two different ways. In such "dual-bunching" logics, the antecedents of sequents are not multisets, but rather finer-grained structures specified using two different punctuation marks to indicate this twofold "bunching" of assumptions (Read, 1988; Restall, 2000).

The first kind of bunching will be indicated here for expository convenience by the comma (though the semicolon is more standard). It is this kind that is used in formulating all the structure-dependent operational rules. These will include unchanged version of $\rightarrow$-I, $\rightarrow$-E, VD, and $\otimes$-I, as well as a generalized version of $\otimes$-E, where $\Delta(\alpha, \beta)$ stands for any structure of which α, β is a substructure:[33]

$$(\otimes\text{-E}_{db})\,\frac{\Gamma \vdash \alpha \otimes \beta \qquad \Delta(\alpha, \beta) \vdash \gamma}{\Delta(\Gamma) \vdash \gamma}\,.$$

In dual-bunching logics, one or more the standard structural rules is rejected for the comma: SContr, SWeak or SExch.[34] However, these logics introduce a second kind of bunching of assumptions, which we will indicate using the colon. This "extensional" bunching *obeys all the standard structural rules*:

$$(\mathsf{eSContr})\,\frac{\Gamma(\alpha : \alpha) \vdash \beta}{\Gamma(\alpha) \vdash \beta}\ (\mathsf{eSWeak})\,\frac{\Gamma(\alpha) \vdash \gamma}{\Gamma(\beta : \alpha) \vdash \gamma}\ (\ \mathsf{eSExch})\,\frac{\Gamma(\alpha : \beta) \vdash \gamma}{\Gamma(\beta : \alpha) \vdash \gamma}\,.$$

Unlike the comma, the colon need not get mentioned in operational rules for any connective. Nor does it get mentioned in the following generalized version of the cut rule:

[32] Zardini's proof makes no use of SWeak. Field's own reasoning amounts to a special case of Zardini's proof, for the case in which we are considering the truth-preservingness of a single-conclusion argument and employ no side assumptions.

[33] For definitions, see Restall (2000, pp. 19-20). In Gentzen calculus formulations, $\otimes$-E$_{db}$ is replaced by $\otimes$-L. Gentzen calculi of this type were developed independently for fragments of relevant logics by Minc (1976) and by Dunn, whose version appears in Anderson and Belnap (1975, §28.5). For natural deduction formulations, see Read (1988), Slaney (1990) and O'Hearn and Pym (1999), whose use of the comma we follow.

[34] Rather than rejecting the structural rule of *associativity*, we are avoiding the need for such a rule by allowing our comma to retain its variable polyadicity.

$$(\mathsf{Cut}_{db})\ \frac{\Gamma \vdash \alpha \qquad \Delta(\alpha) \vdash \beta}{\Delta(\Gamma) \vdash \beta}$$

Just as in the case of multiset-based logics, rejecting SContr for the comma suffices to block the above derivations of c-Curry and v-Curry. Moreover, the Validity Argument is still vindicated. That is because we retain $\otimes$-L, now generalizable to

$$(\otimes\text{-L}_{db})\ \frac{\Gamma(\alpha_1, \alpha_2) \vdash \beta}{\Gamma(\alpha_1 \otimes \alpha_2) \vdash \beta}\ .$$

Thus the reasoning challenged by Field goes through unchanged, provided the conclusion is formulated using the structural comma together with the multiplicative $\otimes$ as the premise-aggregating connective. Construed this way, the Validity Argument's "only if" direction establishes that $\alpha_1, ..., \alpha_n \vdash_T \beta$ only if $\vdash_T Tr(\ulcorner \alpha_1 \urcorner) \otimes ... \otimes Tr(\ulcorner \alpha_n \urcorner) \rightarrow \beta$. However, in the context of a dual-bunching logic, a parallel result now holds as well for the connective &, known in this structural context as "extensional" conjunction. This is because the fact that the colon obeys eSContr allows us to replicate the above derivation of &-L, yielding

$$(\&\text{-L}_{db})\ \frac{\Gamma(\alpha_1 : \alpha_2) \vdash \beta}{\Gamma(\alpha_1 \,\&\, \alpha_2) \vdash \beta}\ .$$

Accordingly, the Validity Argument also goes through when the conclusion is formulated using the structural colon together with & as the premise-aggregating connective. Thus construed, it establishes that $\alpha_1 : ... : \alpha_n \vdash_T \beta$ only if $\vdash_T Tr(\ulcorner \alpha_1 \urcorner) \,\&\, ... \,\&\, Tr(\ulcorner \alpha_n \urcorner) \rightarrow \beta$.[35]

There are thus at least two general ways one to vindicate the Validity Argument for VTP by rejecting SContr: one can use a multiset-based logic with multiplicative conjunction as premise-aggregating connective, or a dual-bunching logic. Moreover, versions of both approaches are known to make possible a naïve theory of truth—either a consistent paracomplete theory or a nontrivial paraconsistent theory.[36] We will return to the difference between the two approaches in the next two sections. For now, we simply note that the two approaches yield logics that conflict over the fragment of the language whose only connectives are & and the

[35]The point extends naturally to cases in which the assumptions are aggregated using both kinds of structure. For instance, $\alpha_1 : (\alpha_2,\ \alpha_3) \vdash_T \beta$ only if $\vdash_T Tr(\ulcorner \alpha_1 \urcorner) \,\&\, (Tr(\ulcorner \alpha_1 \urcorner) \otimes Tr(\ulcorner \alpha_n \urcorner)) \rightarrow \beta$.

[36]Most work on this issue has concerned the closely parallel case of a naïve set theory featuring an unrestricted axiom of comprehension. For proofs of the consistency or nontriviality of unrestricted comprehension in some "weak relevant logics" that can be specified via dual-bunching natural deduction, see Brady (1983, 1989, 2006). For applications of Brady's techniques to naïve truth-theory, see Priest (1991) and Beall (2009), which do not however consider natural deduction systems. As for multiset-based logics, the consistency of unrestricted comprehension in an affine logic was shown by V. Grishin in 1974: see Došen (1993). For the consistency of a naiv̈e truth theory based on an affine logic, see Zardini (2011).

corresponding disjunction $\vee$. Recall that these connectives' (structure-independent) rules *don't even mention* the nonstandard comma structure. It follows immediately that on the dual-bunching approach, whose colon structure obeys all the standard structural rules, the *single-premise* validities of this fragment will be exactly those of the corresponding fragment of classical logic. This stands in contrast to the conjunctive/disjunctive fragment of additive or multiplicative linear logic. On the additive side, we notoriously lose *Distribution* (Belnap, 1993): $\alpha \mathbin{\&} (\beta \vee \gamma) \vdash (\alpha \mathbin{\&} \beta) \vee (\alpha \mathbin{\&} \gamma)$. On the multiplicative side, we lose *Simplification*: $\alpha \otimes \beta \vdash \alpha$. Adding the rule SWeak, as Zardini proposes, restores the latter. As we will see, however, we still lose $\alpha \vdash \alpha \otimes \alpha$.[37]

3.3 From VTP to absurdity via the *Modus Ponens* axiom

We now turn to the objection that VTP entails the *Modus Ponens* axiom and thus absurdity via c-Curry reasoning. Using a generic premise-aggregating connective, we can state, respectively, the validity of *Modus Ponens* and the *Modus Ponens* axiom as follows:

$$(\mathsf{VMP}_{\odot})\ Val(\ulcorner(\alpha \to \beta) \odot \alpha\urcorner, \ulcorner\beta\urcorner).$$

$$(\mathsf{MPA}_{\odot})\ (\alpha \to \beta) \odot \alpha \to \beta.$$

In §1.2 we saw that VTP, when expressed in the objection language, entails

$$(\mathsf{V1})\ Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner) \to (\alpha \to \beta).$$

It follows that if our naïve semantic theory implies $\mathsf{VMP}_{\odot}$, then it also implies $\mathsf{MPA}_{\odot}$. Moreover, in view of VP and VD, the semantic theory will imply $\mathsf{VMP}_{\odot}$ just in case our contraction-free underlying logic gives us $(\alpha \to \beta) \odot \alpha \vdash \beta$. Thus we need to answer two questions:

[37] It would be possible to maintain, within a dual-bunching logic, a connective $\&_A$ that behaves more like the "additive" conjunction and disjunction of a multiset-based logic, for instance in failing to validate Distribution over the corresponding $\vee_A$. To achieve this, replace &-E1 and &-E2 with

$$(\&_A\text{-E1})\ \frac{\Gamma, \alpha \vdash \gamma \qquad \Delta \vdash \alpha \mathbin{\&_A} \beta}{\Gamma, \Delta \vdash \gamma} \qquad (\&_A\text{-E2})\ \frac{\Gamma, \beta \vdash \gamma \qquad \Delta \vdash \alpha \mathbin{\&_A} \beta}{\Gamma, \Delta \vdash \gamma}.$$

By contrast, in the presence of Cut_{db}, our original &-E1 and &-E2 have the same "extensional" effect as the rules

$$(\&\text{-E1}_{db})\ \frac{\Gamma(\alpha) \vdash \gamma \qquad \Delta \vdash \alpha \mathbin{\&_A} \beta}{\Gamma(\Delta) \vdash \gamma} \qquad (\&\text{-E2}_{db})\ \frac{\Gamma(\beta) \vdash \gamma \qquad \Delta \vdash \alpha \mathbin{\&_A} \beta}{\Gamma(\Delta) \vdash \gamma}.$$

We thank Dave Ripley for bringing this to our attention.

(1) If we reject SContr, will we still have $(\alpha \to \beta) \odot \alpha \vdash \beta$?

(2) If we reject SContr, will $\mathsf{MPA}_{\odot}$ still yield absurdity?

As should be expected, the answers to these questions will vary depending on which connective we employ as our $\odot$.[38]

When we use the additive & of a logic without SContr, the answer to (1) is negative (Restall, 1994, pp. 35-6). It should help to display how SContr is involved in the usual derivation:

$$\dfrac{\dfrac{\dfrac{(\alpha \to \beta) \,\&\, \alpha \vdash (\alpha \to \beta) \,\&\, \alpha}{(\alpha \to \beta) \,\&\, \alpha \vdash \alpha \to \beta}\,\&\text{-E} \qquad \dfrac{(\alpha \to \beta) \,\&\, \alpha \vdash (\alpha \to \beta) \,\&\, \alpha}{(\alpha \to \beta) \,\&\, \alpha \vdash \alpha}\,\&\text{-E}}{(\alpha \to \beta) \,\&\, \alpha, (\alpha \to \beta) \,\&\, \alpha \vdash \beta}\,\to\text{-E}}{(\alpha \to \beta) \,\&\, \alpha \vdash \beta}\,\mathsf{SContr}$$

On this construal, then, then the objection to VTP fails, since that thesis will not imply $\mathsf{MPA}_{\,\&}$.

But the objection fails as well when we use the the multiplicative $\otimes$. This time, the answer to (1) is affirmative:

$$\dfrac{\dfrac{\alpha \to \beta \vdash \alpha \to \beta \qquad \alpha \vdash \alpha}{\alpha \to \alpha, \alpha \vdash \beta}\,\to\text{-E} \qquad (\alpha \to \beta) \otimes \alpha \vdash (\alpha \to \beta) \otimes \alpha}{(\alpha \to \beta) \otimes \alpha \vdash \beta}\,\otimes\text{-E}$$

However, now the answer to (2) is negative. That is because, as already noted in Meyer et al. (1979), the above argument from $\mathsf{MPA}_{\odot}$ to absurdity depends essentially on the equivalence $\vdash \alpha \leftrightarrow \alpha \odot \alpha$, specifically its left-to-right direction. But when we use multiplicative conjunction, we lose $\vdash \alpha \to \alpha \otimes \alpha$ (Zardini, 2011). Again, notice how SContr is involved in the usual derivation:

[38] According to the theory proposed by Ripley (2011) based on Cobreros et al. (2011), which is "substructural" only in rejecting Cut, the argument against VTP we are considering in this section fails because MPA fails to yield absurdity. This is because the argument's final step from $\vdash_T \kappa \to \bot$ and $\vdash_T \kappa$ to $\vdash_T \bot$ fails. In Ripley's Gentzen calculus, the rule $\to$-E is inadmissible in the absence of Cut. Indeed, Ripley holds (p.c) that $\to$-E shouldn't be regarded as fundamental to the logic of a detaching conditional, as it covertly builds in extraneous transitivity in comparison with the Gentzen-style rule

$$(\to\text{-L})\ \dfrac{\Gamma \vdash \alpha \qquad \Delta, \beta \vdash \gamma}{\Delta, \alpha \to \beta, \Gamma \vdash \gamma}\ .$$

To this, defenders of $\to$-E may reply that *each of* $\to$-E and $\to$-L builds in transitivity in comparison with the other rule, and in comparison with $\alpha \to \beta, \alpha \vdash \beta$. It is true, as Ripley shows, that the transitivity built in by $\to$-E (which, given $\to$-I, yields Cut) can be blamed for paradox. But in view of the option of blaming paradox on SContr instead, this won't suffice to show that $\to$-L is a more fundamental rule than $\to$-E.

$$\dfrac{\dfrac{\dfrac{\alpha \to \alpha \qquad \alpha \to \alpha}{\alpha, \alpha \vdash \alpha \otimes \alpha} \to\text{-E}}{\alpha \vdash \alpha \otimes \alpha} \mathsf{SContr}}{\vdash \alpha \to \alpha \otimes \alpha} \to\text{-I}$$

We can now summarize our response to those who object to VTP based on the argument involving the *Modus Ponens* axiom. To derive absurdity from the claim that valid arguments preserve truth, the objection presupposes *both* that $(\alpha \to \beta) \odot \alpha \vdash \beta$ and that $\vdash \alpha \to \alpha \odot \alpha$. Yet one or the other fails for each of the premise-aggregating connectives.

At this point, a critic of VTP might object that the response just given is at best incomplete. We have shown that the argument from VTP to absurdity fails, in the absence of SContr, when either of the connectives we have described is used to aggregate the premises of *Modus Ponens*. Still, the critic insists, our task remains that of explaining why the argument fails when our *ordinary notion of conjunction* is used as premise-aggregator. After all, both the single-premise *Modus Ponens* rule $(\alpha \to \beta) \odot \alpha \vdash \beta$ and the *Idempotence* axiom $\vdash \alpha \leftrightarrow \alpha \odot \alpha$ appear to hold for ordinary conjunction. If we are to avoid absurdity in the presence of a naïve theory of truth, we have seen, at least one of these appearances must be mistaken. The unaddressed challenge is to explain which.

Zardini (2011, 2012) has argued that the single-premise *Modus Ponens* clearly holds for our "informal notion of conjunction," whence it is Idempotence that must instead be rejected. Accordingly, he holds that our informal notion is best captured by the multiplicative connective $\otimes$ of an affine logic—where the presence of SWeak guarantees such ordinary features as Simplification $\vdash \alpha \otimes \beta \to \alpha$. Yet, as he recognizes, someone else might argue that Idempotence clearly holds for ordinary conjunction. More generally, it might be held that the usual lattice properties are essential to our ordinary 'and', whence from $\alpha \vdash \beta$ and $\alpha \vdash \gamma$ it must follow that $\alpha \vdash (\beta \textit{ and } \gamma)$, even in the case where $\alpha = \beta = \gamma$. In that case, it would be the single-premise *Modus Ponens* rule that must be rejected.

We do not propose to settle this dispute about our informal notion of conjunction, or examine whether there is a univocal such notion (for a contrary claim, see Mares and Paoli, 2012). Instead, we now wish to explain how the dispute is affected by the availability of dual-bunching logics, which Zardini doesn't consider. In a multiset-based logic without SContr, we have seen, the additive connective & violates the rule

$$(\odot\text{-L})\ \frac{\Gamma, \alpha_1, \alpha_2 \vdash \beta}{\Gamma, \alpha_1 \odot \alpha_2 \vdash \beta}\,.$$

Indeed, we have a counterexample in the failure of $\alpha \to \beta, \alpha \vdash \beta$ to yield $(\alpha \to \beta)\ \&\ \alpha \vdash \beta$. But Zardini insists that $\odot$-L is non-negotiable for ordinary conjunction, which he says is the connective we use to make explicit "how premises are combined in a multi-premise argument" (Zardini, 2012). This is the chief reason why Zardini concludes that $\otimes$ has a stronger claim than & to represent our informal notion of conjunction.[39]

But once dual-bunching logics are available, matters get more complicated. As explained above, in such logics we have both &-L_{db} and $\otimes$-L_{db}. The additive connective & corresponds to one mode of premise-aggregation, marked by our colon, while the multiplicative $\otimes$ corresponds to another mode of premise-aggregation, marked by our comma (see esp. Read, 1988). According to dual-bunching logics, &-L_{db} with its *colon* structure does not yield the single-premise *Modus Ponens* rule, since it is only the *comma* structure that appears in the $\rightarrow$-I and $\rightarrow$-E rules. Yet in view of &-L_{db}, the connective & has a claim, by Zardini's own lights, to represent our informal notion of conjunction, provided the structural comma can be seen as standing for one mode in which premises are combined in a multi-premise argument. Once dual-bunching logics are under consideration, then, it is less clear that Zardini's view on which ordinary conjunction is multiplicative and obeys the single-premise *Modus Ponens* rule holds any advantage over the alternative view on which ordinary conjunction is additive and satisfies Idempotence.

In this section, we have shown that the standard argument from VTP to absurdity breaks down in substructural theories which do not validate SContr, and have explained how the details of *where* it breaks down depend on which connective of the contraction-free logic we use to represent the conjunction appealed to in the standard argument.

3.4 The Consistency Argument

Let us finally turn to the Consistency Argument, and the resulting challenge to VTP from Gödel's Second Incompleteness Theorem. There are two ways one might respond to this challenge: argue that Gödel's limitative results don't obtain for theories of arithmetic based on contraction-free logics, or argue that the Consistency Argument fails for such logics. Since there are contraction-free theories of arithmetic for which the results hold, we will not rely exclusively on the former strategy.[40]

[39]Ole T. Hjortland (2012) has recently proposed using an affine logic with additive conjunction and disjunction in a revisionary approach to semantic paradox. We take no position here on whether the consideration just rehearsed poses a serious problem for that approach.

[40]Restall (1994, ch. 11) shows that that an arithmetic based on the dual-bunching contraction-free logic RWK (which he calls CK) is classical Peano arithmetic, but it is not known whether RWK supports a nontrivial naïve semantic theory in which $Tr(\ulcorner\alpha\urcorner)$ is everywhere intersubstitutable with α (see Hjortland, 2012).

The Consistency Argument requires one to prove, within one's semantic theory T, the following induction step: if *all* conclusions of derivations of length $\leq n$ are true, then *all* conclusions of derivations of length $n+1$ are true. To prove this, it will suffice to prove, for each rule R, that

(TP_R) If *all* the premises of an instance of R are true, then the corresponding instance of the conclusion will be true.[41]

Now consider a rule R such that the theory proves that R has precisely two premises. To establish TP_R we will then need to prove

($\mathsf{TP2}_R$) For all x, y, z such that x and y are the two premises of an instance of R and z its corresponding conclusion: if x is true *and* y is true, then z is true.

But how are we to understand the 'and' in $\mathsf{TP2}_R$?

If the 'all' in TP_R is understood as the standard "lattice-theoretical" or additive quantifier (Paoli, 2005), then $\mathsf{TP2}_R$ will only help establish TP_R provided that 'and' is likewise construed as additive.[42] But when R is the two-premise *Modus Ponens* rule, we have already seen that we don't have any instance of $\vdash_T (\alpha \to \beta) \,\&\, \alpha \to \beta$. Given the substitutivity of equivalents in conjunctions, this means that we don't have any instance of $\vdash_T Tr(\ulcorner\alpha \to \beta)\urcorner \,\&\, Tr(\ulcorner\alpha\urcorner) \to Tr(\ulcorner\beta\urcorner)$ either, whence we can't prove the generalization TP_R. In fact, this is Field's own explanation of how the Consistency Argument breaks down for paracomplete and paraconsistent theories (Field, 2008, pp. 377-8). Unlike Field, we are not interpreting this breakdown as resulting from the failure of VTP. Rather, in our view, the breakdown of the Consistency Argument on the standard interpretation of the quantifier results from the failure of & to serve as premise-aggregator for the two-premise *Modus Ponens* rule.

Perhaps, then, we could rescue the Consistency Argument by interpreting the 'all' in TP_R as some kind of multiplicative quantifier, one that stands to our multiplicative $\otimes$ the way the standard universal quantifier stands to &. Where R is *Modus Ponens*, we should indeed be able to prove $\mathsf{TP2}_R$ with 'and' interpreted as $\otimes$, since

[41] Here we are no longer thinking of natural deduction rules, but rather of the rules of a Hilbert system, rules for generating theorems.

[42] Here is a rough explanation. In the course of deriving TP_R, one will need to establish, under the assumption that (ignoring use-mention issues) a_1, a_2 and b are the respective premises and conclusion of an instance of R, the claim $\forall x(x = a_1 \vee x = a_2 \to Tr(x)) \vdash Tr(b)$. Assuming $\forall$ is lattice-theoretical, this claim will follow from $Tr(a_1) \,\&\, Tr(a_2) \vdash Tr(b)$, whereas it won't follow from $Tr(a_1) \otimes Tr(a_2) \vdash Tr(b)$. For we have $\forall x\phi(x) \vdash \phi(a_1) \,\&\, \phi(a_2)... \,\&\, ...\phi(a_n)$, but not $\forall x\phi(x) \vdash \phi(a_1) \otimes \phi(a_2)... \otimes ...\phi(a_n)$. See Běhounek et al. (2007).

$\otimes$ does serve as premise-aggregator for *Modus Ponens*.[43] If this is to help establish TP_R, however, we would need to know more about the envisioned multiplicative quantifier. Paoli (2005) and Mares and Paoli (2012) note that there is no accepted theory of how such a quantifier should behave. One option is presented by Zardini (2011) in the context of a multiset-based logic. But Zardini's multiplicative quantifier will not serve the purposes of anyone who—unlike Zardini himself—wishes to use the Consistency Argument to criticize VTP. For he characterizes the behavior of the multiplicative quantifier using an ω-rule as (right-)introduction rule. Hence, the semantic theory based on this logic will not be recursively axiomatisable, and will not satisfy the conditions for Gödel's theorem.

As promised, once SContr is restricted, the standard arguments against VTP break down. It follows, then, that such arguments at best support the weaker conclusion that, given the naïve view of truth, *either* VTP *or* SContr must fail. To be sure, rcf theorists, especially Field, are aware of the existence of substructural revisionary approaches. Field dismisses them, though, as "radical," (Field, 2008, p. 10) and as "very desperate measures" that are, ultimately, not needed (Field, 2009a, p. 350). He writes:

> I haven't seen sufficient reason to explore this kind of approach (which I find very hard to get my head around), since I believe we can do quite well without it. ... [Hence] I will take the standard structural rules for granted. (Field, 2008, pp. 10-11; also 283n)

However, while we agree with Field that more work needs to be done to make sense of a failure of SContr, we'd like to stress that giving up VTP is *also* a radical move. Moreover, we hope to have shown that, *pace* Field, it is unclear whether the naïve conception of semantic properties is consistent with the standardly accepted structural rules: as we've argued in §§2.2-3, the revisionary approach to semantic paradox might itself be a sufficient reason for rejecting SContr.

4 Concluding remarks

In this paper, we've argued for two main claims. First, SContr is in tension with natural principles governing some (intuitive enough) notions of validity. Second, the standard arguments against VTP presented in §1 all break down once SContr

[43]Essentially this point is made by Field (2008, p.378-0), albeit with $Tr(\ulcorner \alpha \rightarrow \beta)\urcorner \otimes Tr(\ulcorner \alpha \urcorner) \rightarrow Tr(\ulcorner \beta \urcorner)$ replaced by $Tr(\ulcorner \alpha \rightarrow \beta)\urcorner \rightarrow (Tr(\ulcorner \alpha \urcorner) \rightarrow Tr(\ulcorner \beta \urcorner))$, which is equivalent to the former in the logics we are considering. See also Priest (2010).

is dropped. Rejecting SContr opens up non-classical ways of aggregating together premises—ways which no longer underwrite the arguments against VTP. To be sure, it may be argued instead that the notion of validity that is shown to be paradoxical by the v-Curry Paradox should be rejected as incoherent. Validity, one might think, is interpretational, or purely logical, validity: truth on all uniform interpretations of the non-logical vocabulary. This, however, does not seem in line with the seemingly compelling thought, championed by rcf theorists such as Field (2007, 2008) and Priest (2006b,a), that logical validity is a *species* of a more general notion of validity. Alternatively, it may be contended that paradox-prone notions of validity must be refined, and made less naïve (McGee, 1991). But this, too, we've argued, doesn't seem like a viable option for proponents of the revisionary approach to paradox, who rather recommend revising our theory of logic, while preserving the naïve semantic principles. If neither of these foregoing options is viable, then SContr must be restricted on pain of triviality, and valid arguments can, after all, preserve truth.

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